Quantum Mechanics

1. Quantum State and Hilbert Space

- Solving Problems

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1. **Problem 1**

   If the states $\{|1\rangle, |2\rangle, |3\rangle\}$ form an orthonormal basis and if the operator $\hat{K}$ has the properties

   $$\hat{K}|1\rangle = 2|1\rangle$$
   $$\hat{K}|2\rangle = 3|2\rangle$$
   $$\hat{K}|3\rangle = -6|3\rangle$$

   (a) Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $|1\rangle, |2\rangle, |3\rangle$ basis.

   (b) What is the expectation or average value of $\hat{K}$, defined as $\langle\alpha|\hat{K}|\alpha\rangle$, in the state

   $$|\alpha\rangle = \frac{1}{4}(-3|1\rangle + i\sqrt{6}|2\rangle + |3\rangle)$$

2. **Problem 2**

   The Hamiltonian operator for a two-state system is given by

   $$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

   where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

3. **Problem 3**

   Consider the states $|\psi\rangle = 9i\,|\phi_1\rangle + 2\,|\phi_2\rangle$ and $|\chi\rangle = -\frac{i}{\sqrt{2}}\,|\phi_1\rangle + \frac{1}{\sqrt{2}}\,|\phi_2\rangle$ where the two vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ form a complete and orthonormal basis.

   (a) Calculate the operators $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$. Are they equal?
   item Find the Hermitian conjugates of $|\psi\rangle, |\chi\rangle, |\psi\rangle\langle\chi|$, and $|\chi\rangle\langle\psi|$

   (b) Calculate $\mathrm{Tr}(|\psi\rangle\langle\chi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\psi|)$. Are they equal?

   (c) Calculate $|\psi\rangle\langle\psi|$ and $\chi\rangle\langle\chi|$ and the traces $\mathrm{Tr}(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\chi|)$. Are they projection operators?

4. **Problem 4**

   Consider a system whose Hamiltonian is given by

   $$\hat{H} = \alpha\left(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|\right)$$

   where α is a real number having the dimensions of energy and $|\phi_1\rangle, |\phi_2\rangle$ are normalized eigenstates of a Hermitian operator $\hat{A}$ that has no degenerate eigenvalues.

(a) Is $\hat{H}$ a projection operator? What about $\alpha^{-2}\hat{H}^2$?

(b) Show that $|\phi_1\rangle$ and $|\phi_2\rangle$ are not eigenstates of $\hat{H}$.

(c) Calculate the commutators $\left[\hat{H}, |\phi_1\rangle\langle\phi_1|\right]$ and $\left[\hat{H}, |\phi_2\rangle\langle\phi_2|\right]$ then find the relation that may exist between them.

(d) Find the normalized eigenstates of $\hat{H}$ and their corresponding energy eigenvalues.

(e) Assuming that $|\phi_1\rangle$ and $|\phi_2\rangle$ form a complete and orthonormal basis, find the matrix representing $\hat{H}$ in the basis. Find the eigenvalues and eigenvectors of the matrix and compare the results with those derived in (d).

5. **Problem 5**

Construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that

$$\mathbf{S}\cdot\hat{\mathbf{n}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \left(\frac{\hbar}{2}\right)|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$$

where $\mathbf{S} = \{S_x, S_y, S_z\}$ are the spin-1/2 matrices

$$S_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|),$$
$$S_y = \frac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|),$$
$$S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|)$$

and $\hat{\mathbf{n}}$ is characterised by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$.

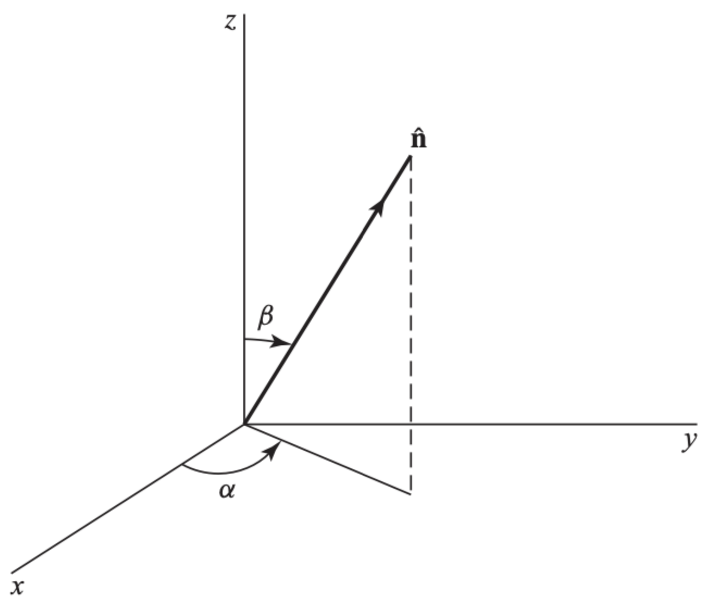

Figure 1: